

HOW FERMAT FOUND EXTREMA ¹A. Skopenkov ²

Abstract. We present a short elementary proof of the well-known criterion for a cubic polynomial to have three real roots. The proof is based on Fermat's approach to calculus for polynomials, and illustrates the idea of a derivative rigorously but without technical ε - δ language. The note is accessible to high-school students.

In elementary school one learns the following.

Let a, b be real numbers. Then the following conditions are equivalent:

- *there are real numbers x, y such that $a = x + y$ and $b = xy$;*
- *the equation $t^2 - at + b = 0$ has a real root;*
- $4b - a^2 \leq 0$.

In this note we illustrate the idea of a derivative by proving the following well-known generalization (which might be proved by Fermat or earlier).

Theorem 1. *Let a, b, c be real numbers. There are real numbers x, y, z such that*

$$a = x + y + z, \quad b = xy + yz + zx \quad \text{and} \quad c = xyz$$

if and only if

$$4 \left(b - \frac{a^2}{3} \right)^3 + \left(c - \frac{ab}{3} + \frac{2a^3}{27} \right)^2 \leq 0.$$

This result is very useful; for recent applications to elementary inequalities see [DFMS] and references therein. Not only this result, but our simple exposition of its proof is hopefully interesting. We do not use the notion of a derivative. However, we illustrate the idea of a derivative rigorously but without technical ε - δ language, cf. [ZSS, §8.1, §8.2]. So this exposition might be useful for introductory courses on analysis. Unfortunately, this exposition is not so well-known, cf. [Po84]. ³ For development of the 'graphs of functions' idea see [FT, Go10, Ta88] (in particular, one can apply this idea to the *pqr*-lemmas of [DFMS]).

There is an alternative proof using complex numbers and calculation of the *discriminant* of cubic polynomial in terms of coefficients, see e.g. [DFMS, §2 and solutions of problems 6-22]. Although that proof is longer, it illustrates another interesting important ideas. In [Ta88] a geometric interpretation (but not a proof) of Theorem 1 is presented.

A reformulation of Theorem 1 in terms of cubic polynomials. By Vieta Theorem (i.e. by direct calculation) the condition on x, y, z is equivalent to the following equality of polynomials:

$$t^3 - at^2 + bt - c = (t - x)(t - y)(t - z).$$

A reduction of Theorem 1 to the case $a = 0$. Take $u := t - \frac{a}{3}$. Then $t = u + \frac{a}{3}$, so

$$t^3 - at^2 + bt - c = u^3 + \left(b - \frac{a^2}{3} \right) u - \left(c - \frac{ab}{3} + \frac{2a^3}{27} \right).$$

Thus it suffices to prove Theorem 1 for $a = 0$.

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³E.g. in June 2016 it was suggested to add this proof to [DFMS] in order to make Theorem 1 and its applications more accessible. The authors found this proof 'too complicated'.

Proof of Theorem 1 for $a = 0, b \geq 0$. Assume that the required x, y, z exist. Since $b \geq 0$, the function $t^3 + bt - c$ is strictly increasing as a sum of increasing functions, one of them strictly increasing. Then the equation $t^3 + bt - c = 0$ has at most one real root. This and $a = 0$ imply that $x = y = z = 0$, hence $4b^3 + 27c^2 = 0$.

If $4b^3 + 27c^2 \leq 0$, then $b = c = 0$, so we can take $x = y = z = 0$. \square

Heuristic considerations for investigation of the function $f(t) := t^3 - 3t$ (formally they are not used in the proof). The function f is strictly increasing on an interval if $f(t_1) > f(t_2)$ for each different $t_1 > t_2$ from the interval. This is equivalent to $\varphi(t_1, t_2) > 0$ for each different t_1, t_2 from the interval, where

$$(*) \quad \varphi(t_1, t_2) := \frac{f(t_1) - f(t_2)}{t_1 - t_2} = t_1^2 + t_1 t_2 + t_2^2 - 3.$$

If these conditions hold for each two ‘close’ t_1, t_2 , then by transitivity they hold for each two t_1, t_2 . So we guess that the ‘boundary points’ of the intervals on which f is monotonous are the roots of the equation $t^2 + tt + t^2 - 3 = 0$. These roots are ± 1 . (This is analogous to [Be88]; considering a simple example before general method makes the method more accessible.)

Lemma 2. *The function $f(t) := t^3 - 3t$ is strictly increasing on $(-\infty, -1]$, is strictly decreasing on $[-1, 1]$ and is strictly increasing on $[1, +\infty)$.*

Proof. Denote $\varphi(t_1, t_2)$ by the formula (*). Then $\varphi(t_1, t_2) > 0$ for each different $t_1, t_2 \geq 1$. Hence $f(t)$ strictly increases on $[1, +\infty)$. Analogously the other two statements are proved. \square

Proof of Theorem 1 for $a = 0, b < 0$. Taking $u := t\sqrt{-b/3}$ we may assume that $b = -3$.

First assume that the required x, y, z exist. Since $-3 = b < 0 = a$, the case $x = y = z$ is impossible, so the equation $f(t) = t^3 - 3t - c = 0$ has at least two real roots.

Denote $f(t) := t^3 - 3t - c = t(t^2 - 3) - c$. Then $t_+ := 1 + \max\{3, |c|\} > 1$ and $f(t_+) > 0$. Analogously, there is $t_- < -1$ such that $f(t_-) < 0$. So by Lemma 2 and the Intermediate Value Theorem the equation $f(t) = 0$ has at least two real roots if and only if $f(-1)$ and $f(1)$ ‘have different signs’, i.e. $f(-1)f(1) \leq 0$. We have $-f(t) = c - t(t^2 - 3)$, so

$$f(-1)f(1) = (c + 2 \cdot 1)(c - 2 \cdot 1) = c^2 - 4 = (4b^3 + 27c^2)/27.$$

Now assume that $4b^3 + 27c^2 = 27(c^2 - 4) \leq 0$. If $c^2 < 4$, the above paragraph shows that the equation $f(t) = 0$ has three real roots, and we take them as x, y, z . If $c^2 = 4$, we take $x = y = -\operatorname{sgn} c$ and $z = 2\operatorname{sgn} c$ (it should be clear how to guess these formulas). \square

Exercise. Find maximal intervals on which the following function is strictly increasing (decreasing):

$$(a) f(t) = t^4 - 4t. \quad (b) f(t) = t^4 - 12t^3 + 22t^2 - 24t + 10.$$

In a forthcoming note joint with A. Sgibnev we plan to present exposition accessible to wider audience.

References

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